Wave localisation in periodic media.

Yannick DE BRUIJN

Joint work with E.O. Hiltunen, H. Ammari, C. Thalhammer and P. Liu

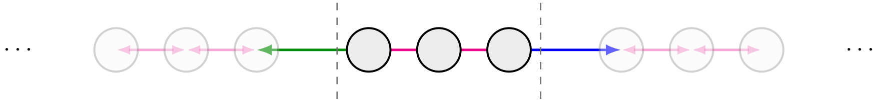
University of Oslo

22 July 2026

Table of Contents

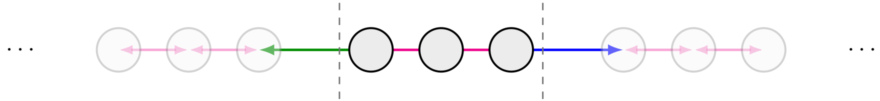
Table of Contents

- **Nearest Neighbour interaction:**



Tight Binding Hamiltonian

- **Nearest Neighbour interaction:**

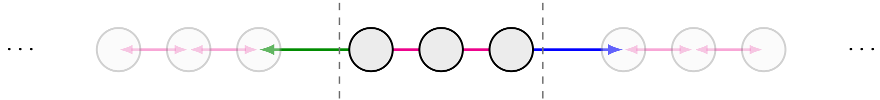


- **Tight-Binding Hamiltonian:**

$$H(\alpha) = H_{-1}e^{-\mathrm{i}\alpha} + H_0 + H_1 e^{\mathrm{i}\alpha}$$

Tight Binding Hamiltonian

- **Nearest Neighbour interaction:**



- **Tight-Binding Hamiltonian:**

$$H(\alpha) = H_{-1}e^{-\mathrm{i}\alpha} + H_0 + H_1 e^{\mathrm{i}\alpha}$$

  - H_{-1} and H_1 coupling between unit cells.
  - H_0 coupling in unit cell limited to nearest neighbours.

Tridiagonal k-Toeplitz operators.

Example: 2-Toeplitz operator:

$$\mathbf{T}(\mathbf{f}) := \begin{pmatrix} a_1 & b_1 & 0 & 0 & & & \\ c_1 & a_2 & b_2 & 0 & & & \\ 0 & c_2 & a_1 & b_1 & 0 & 0 & \\ 0 & 0 & c_1 & a_2 & b_2 & 0 & \\ & & 0 & c_k & \ddots & & \ddots \\ & & 0 & 0 & & & \\ & & & & \ddots & & \end{pmatrix} = \begin{pmatrix} \mathbf{A}_0 & \mathbf{A}_{-1} & & \\ \mathbf{A}_1 & \mathbf{A}_0 & \mathbf{A}_{-1} & \\ & \mathbf{A}_1 & \ddots & \ddots \\ & & \ddots & \end{pmatrix}$$

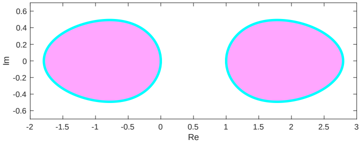
- **Symbol function (Bloch Hamiltonian):**

$$\begin{aligned} f : \mathbb{C} &\to \mathbb{C}^{2\times 2} \\ z &\mapsto \mathbf{A}_{-1} z^{-1} + \mathbf{A}_0 + \mathbf{A}_1 z = \begin{pmatrix} a_1 & b_1 + c_2 z \\ c_1 + b_2 z^{-1} & a_2 \end{pmatrix}. \end{aligned}$$

Spectrum of k-Toeplitz operators

- Let us define the following sets:

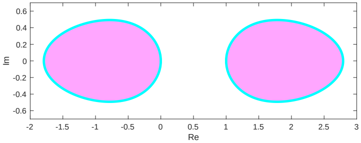
$$\sigma_{\det}(f) := \{\lambda \in \mathbb{C} : \det\big(f(z) - \lambda\big) = 0,\ \exists z \in \mathbb{T}\},$$
$$\sigma_{\mathrm{wind}}(f) := \{\lambda \in \mathbb{C} \setminus \sigma_{\det}(f) : \mathrm{wind}\big(\det\big(f(\mathbb{T}) - \lambda\big), 0\big) \neq 0\}.$$



Spectrum of k-Toeplitz operators

- Let us define the following sets:

$$\sigma_{\det}(f) := \{\lambda \in \mathbb{C} : \det\big(f(z) - \lambda\big) = 0,\ \exists z \in \mathbb{T}\},$$
$$\sigma_{\mathrm{wind}}(f) := \{\lambda \in \mathbb{C} \setminus \sigma_{\det}(f) : \mathrm{wind}\big(\det\big(f(\mathbb{T}) - \lambda\big), 0\big) \neq 0\}.$$



Theorem (dB, Ammari, Liu, Thalhammer)

Let $f \in \mathbb{C}^{k\times k}(\mathbb{T})$ be the symbol of a tridiagonal k-Toeplitz operator $T(f)$ it holds that

$$\sigma_{\det}(f) \cup \sigma_{\mathrm{wind}}(f) \subseteq \sigma\big(T(f)\big) \subseteq \sigma_{\det}(f) \cup \sigma_{\mathrm{wind}}(f) \cup \sigma(\mathbf{B}_0),$$

$\mathbf{B_0} \in \mathbb{R}^{k-1\times k-1}$ *is the principal submatrix of* $\mathbf{A}_0$.

Exact similarity of Toeplitz matrices and open limit

- Let $\mathbf{T}_n(f) \in \mathbb{R}^{n\times n}$ be finite Toeplitz matrix.

Exact similarity of Toeplitz matrices and open limit

- Let $\mathbf{T}_n(f) \in \mathbb{R}^{n\times n}$ be finite Toeplitz matrix.
- The similarity transform:

$$\sigma\big(\mathbf{T}_n(f(z))\big) = \sigma\big(\mathbf{T}_n(f(rz))\big), \quad r > 0,\ z \in \mathbb{T}.$$

Exact similarity of Toeplitz matrices and open limit

- Let $\mathbf{T}_n(f) \in \mathbb{R}^{n\times n}$ be finite Toeplitz matrix.
- The similarity transform:

$$\sigma\big(\mathbf{T}_n(f(z))\big) = \sigma\big(\mathbf{T}_n(f(rz))\big), \quad r > 0, \; z \in \mathbb{T}.$$

Theorem (Schmidt-Spitzer, 1960)

Let f be the symbol function, then the Hausdorff limit is given by

$$\lim_{n\to\infty} \sigma\big(\mathbf{T}_n(f)\big) = \bigcap_{r\in(0,\infty)} \sigma\Big(\mathbf{T}\big(f(r\mathbb{T})\big)\Big).$$

Exact similarity of Toeplitz matrices and open limit

- Let $\mathbf{T}_n(f) \in \mathbb{R}^{n\times n}$ be finite Toeplitz matrix.
- The similarity transform:

$$\sigma\big(\mathbf{T}_n(f(z))\big) = \sigma\big(\mathbf{T}_n(f(rz))\big), \quad r > 0, \; z \in \mathbb{T}.$$

Theorem (Schmidt-Spitzer, 1960)

Let f be the symbol function, then the Hausdorff limit is given by

$$\lim_{n\to\infty} \sigma\big(\mathbf{T}_n(f)\big) = \bigcap_{r\in(0,\infty)} \sigma\Big(\mathbf{T}\big(f(r\mathbb{T})\big)\Big).$$

Particular for Tridiagonal k-Toeplitz case:

- Exists a unique $\tilde{r} > 0$ such that:

$$\lim_{n\to\infty} \sigma\big(\mathbf{T}_n(f)\big) = \sigma\Big(\mathbf{T}\big(f(\tilde{r}z)\big)\Big).$$

Exact similarity of Toeplitz matrices and open limit

- Let $\mathbf{T}_n(f) \in \mathbb{R}^{n\times n}$ be finite Toeplitz matrix.
- The similarity transform:

$$\sigma\big(\mathbf{T}_n(f(z))\big) = \sigma\big(\mathbf{T}_n(f(rz))\big), \quad r > 0,\ z \in \mathbb{T}.$$

Theorem (Schmidt-Spitzer, 1960)

Let f be the symbol function, then the Hausdorff limit is given by

$$\lim_{n\to\infty} \sigma\big(\mathbf{T}_n(f)\big) = \bigcap_{r\in(0,\infty)} \sigma\Big(\mathbf{T}\big(f(r\mathbb{T})\big)\Big).$$

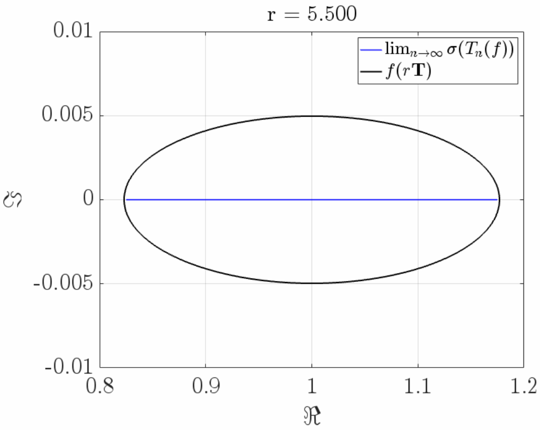
Particular for Tridiagonal k-Toeplitz case:

- Exists a unique $\tilde{r} > 0$ such that:

$$\lim_{n\to\infty} \sigma\big(\mathbf{T}_n(f)\big) = \sigma\Big(\mathbf{T}\big(f(\tilde{r}z)\big)\Big).$$

- $\log(\tilde{r})$ is the exponential decay rate of the eigenvectors of $\mathbf{T}_n(f)$.

Scaling of the symbol function



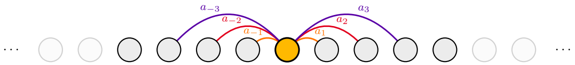
- There exists a unique $\tilde{r} \approx 5.658$ which collapses the symbol function[1].

[1] Ammari et al "Generalised Brillouin Zone for Non-Reciprocal Systems"

Table of Contents

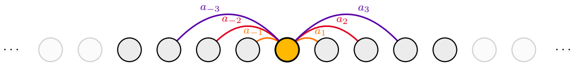
Long range coupling (m-banded Toeplitz)

- Interaction with m-nearest neighbours.



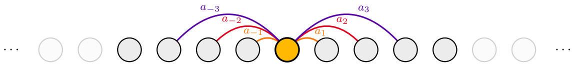
Long range coupling (m-banded Toeplitz)

- Interaction with m-nearest neighbours.
- Leads to m-banded Toeplitz matrix.



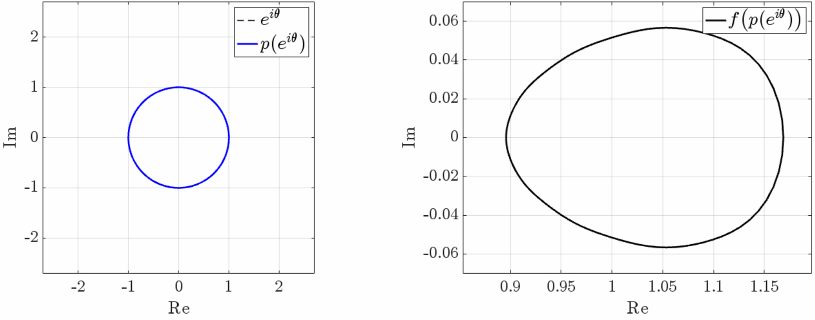
Long range coupling (m-banded Toeplitz)

- Interaction with m-nearest neighbours.
- Leads to m-banded Toeplitz matrix.



$$\mathbf{T}_n(f) = \begin{pmatrix} a_0 & a_{-1} & \cdots & a_{-m} & & \\ a_1 & a_0 & \ddots & & \ddots & \\ \vdots & \ddots & \ddots & \ddots & & a_{-m} \\ a_m & & \ddots & \ddots & \ddots & \vdots \\ & \ddots & & \ddots & \ddots & a_{-1} \\ & & a_m & \cdots & a_1 & a_0 \end{pmatrix}, \quad f(z) = \sum_{k=-m}^{m} a_k z^{-k}, \quad a_k \in \mathbb{R},\ z \in \mathbb{T}.$$

Scaling of unit torus in m-banded Toeplitz case



- Contrary to tridiagonal k-Toeplitz, no unique $\tilde{r}$ to uniformly collapse the symbol.

Allow deformations of the torus

Construction:

- Roots to shifted symbol function:

$$z^m\big(f(z)-\lambda\big)=\prod_{i=1}^{2m}\big(z-z_i(\lambda)\big).$$

[2] Schmidt-Spitzer "The Toeplitz Matrices of an Arbitrary Laurent Polynomial"

Allow deformations of the torus

Construction:

- Roots to shifted symbol function:

$$z^m\big(f(z)-\lambda\big) = \prod_{i=1}^{2m}\big(z - z_i(\lambda)\big).$$

- Sort roots in ascending magnitude:

$$|z_1(\lambda)| \leq \cdots \leq |z_m(\lambda)| \leq |z_{m+1}(\lambda)| \leq \cdots \leq |z_{2m}(\lambda)|.$$

[2] Schmidt-Spitzer "The Toeplitz Matrices of an Arbitrary Laurent Polynomial"

Allow deformations of the torus

Construction:

- Roots to shifted symbol function:

$$z^m\big(f(z)-\lambda\big) = \prod_{i=1}^{2m}\big(z - z_i(\lambda)\big).$$

- Sort roots in ascending magnitude:

$$|z_1(\lambda)| \leq \cdots \leq |z_m(\lambda)| \leq |z_{m+1}(\lambda)| \leq \cdots \leq |z_{2m}(\lambda)|.$$

- Define Generalised Brillouin Zone[2] (GBZ) as,

$$\mathbf{Z}(f) := \left\{z_m, z_{m+1} \in \mathbb{C} \ : \ |z_m(\lambda)| = |z_{m+1}(\lambda)|, \text{ for some } \lambda \in \mathbb{R}\right\}.$$

[2] Schmidt-Spitzer "The Toeplitz Matrices of an Arbitrary Laurent Polynomial"

Allow deformations of the torus

Construction:

- Roots to shifted symbol function:

$$z^m\big(f(z)-\lambda\big)=\prod_{i=1}^{2m}\big(z-z_i(\lambda)\big).$$

- Sort roots in ascending magnitude:

$$|z_1(\lambda)|\leq\cdots\leq|z_m(\lambda)|\leq|z_{m+1}(\lambda)|\leq\cdots\leq|z_{2m}(\lambda)|.$$

- Define Generalised Brillouin Zone[2] (GBZ) as,

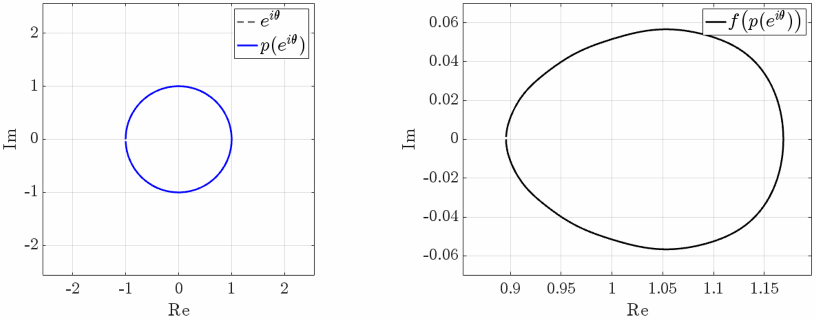
$$\mathbf{Z}(f):=\Big\{z_m,z_{m+1}\in\mathbb{C}\;:\;|z_m(\lambda)|=|z_{m+1}(\lambda)|,\ \text{for some }\lambda\in\mathbb{R}\Big\}.$$

- Assume we can write

$$\mathbf{Z}(f)=p(\mathbb{T}):=\bigcup_{\theta\in[0,\pi]}p(e^{\mathrm{i}\theta})=\text{GBZ}.$$

[2] Schmidt-Spitzer "The Toeplitz Matrices of an Arbitrary Laurent Polynomial"

Deform Unit Circle to GBZ



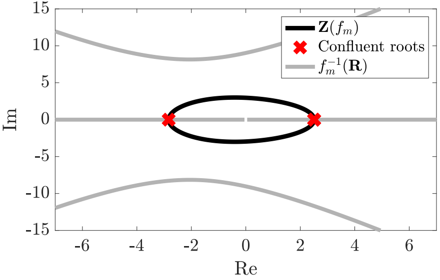
Open vs periodic boundary conditions

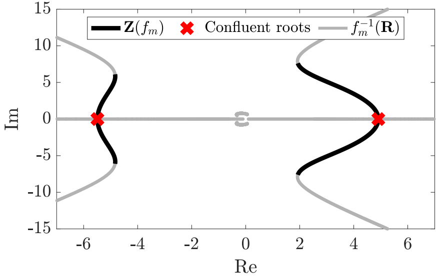
Theorem (dB, Hiltunen)

Suppose that the GBZ, i.e. $\mathbf{Z}(f)$ is spanned by a polar curve $p(\mathbb{T})$, then it holds that

$$\lim_{n\to\infty} \sigma\big(\mathbf{T}_n(f)\big) = \bigcup_{\alpha\in[-\pi,\pi]} f\big(e^{\mathrm{i}(\alpha+\mathrm{i}\beta(\alpha))}\big) \subset \mathbb{R}.$$

Examples:





Conservation of Density of States

- No longer have exact conserved spectrum at the matrix level, i.e.

$$\sigma\big(\mathbf{T}_n(f)\big) \neq \sigma\big(\mathbf{T}_n(f \circ p)\big)$$

Conservation of Density of States

- No longer have exact conserved spectrum at the matrix level, i.e.

$$\sigma\big(\mathbf{T}_n(f)\big) \neq \sigma\big(\mathbf{T}_n(f \circ p)\big)$$

Theorem (dB, Hiltunen)

Suppose that the GBZ is $C^1([0, 2\pi])$, *then*

$$DoS\big(\mathbf{T}_n(f)\big) = DoS\big(\mathbf{T}_n(f \circ p)\big).$$

Conservation of Density of States

- No longer have exact conserved spectrum at the matrix level, i.e.

$$\sigma\big(\mathbf{T}_n(f)\big) \neq \sigma\big(\mathbf{T}_n(f \circ p)\big)$$

Theorem (dB, Hiltunen)

Suppose that the GBZ is $C^1([0, 2\pi])$, then

$$DoS\big(\mathbf{T}_n(f)\big) = DoS\big(\mathbf{T}_n(f \circ p)\big).$$

Empirical measure vs DoS:

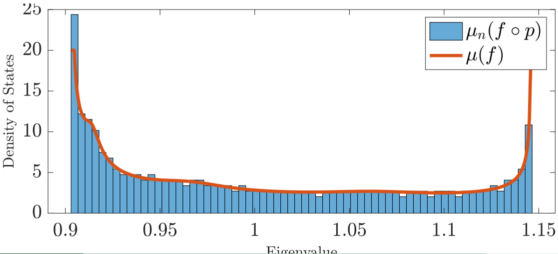


Table of Contents

Applications (Computational)

- Numerically Hausforff distance

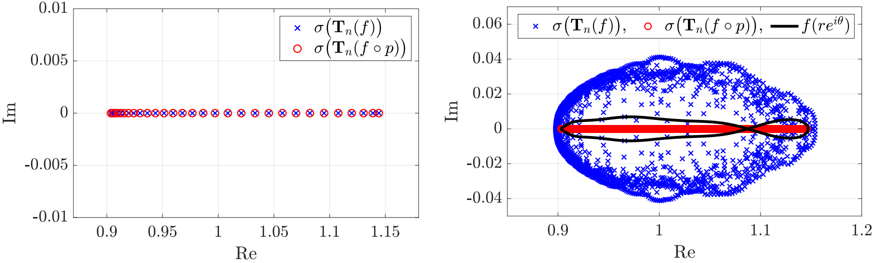
$$d_H\Big(\sigma\big(\mathbf{T}_n(f)\big), \sigma\big(\mathbf{T}_n(f \circ p)\big)\Big) = \mathcal{O}(n^{-2}).$$

Applications (Computational)

- Numerically Hausforff distance

$$d_H\Big(\sigma\big(\mathbf{T}_n(f)\big),\sigma\big(\mathbf{T}_n(f\circ p)\big)\Big) = \mathcal{O}(n^{-2}).$$

- Spectra of large but finite Toeplitz Matrices:

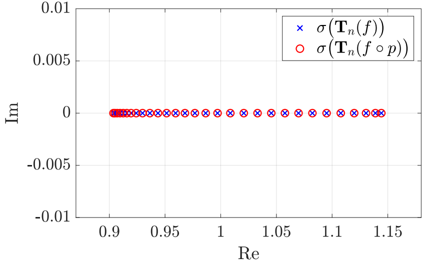


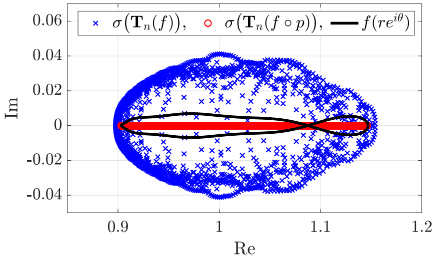
Applications (Computational)

- Numerically Hausforff distance

$$d_H\Big(\sigma\big(\mathbf{T}_n(f)\big), \sigma\big(\mathbf{T}_n(f\circ p)\big)\Big) = \mathcal{O}(n^{-2}).$$

- Spectra of large but finite Toeplitz Matrices:





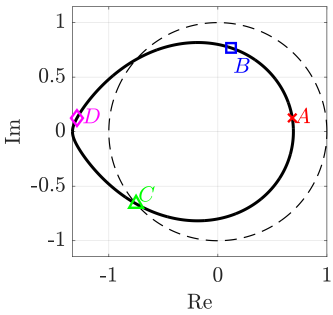
- Precondition (even normalise) a class of m-banded non-Hermitian Toeplitz matrices.

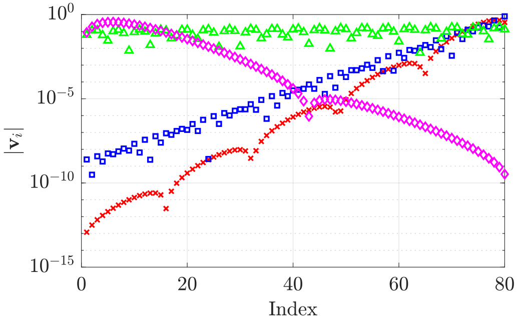
Applications (Condensed matter Physics)

- Predict conservation of energy (polar curve GBZ).

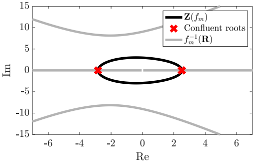
Applications (Condensed matter Physics)

- Predict conservation of energy (polar curve GBZ).
- Predict (frequency specific) eigenmode localisation.

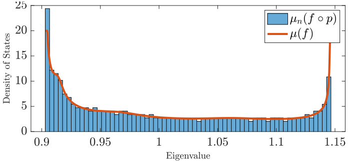




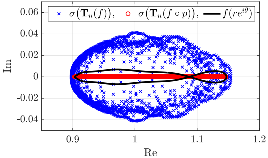
Thank you for your attention.



(a) Reality of open limit.



(b) Density of states.



(c) Numerical Preconditioner.

Main References:

[1] dB, Ammari, Thalhammer, Liu, Barandun: *Spectra and pseudo-spectra of tridiagonal k-Toeplitz matrices and the topological origin of the non-Hermitian skin effect* 10.1088/1751-8121/add5ab

[2] dB, Hiltunen: *Mathematical Foundation for the Generalised Brillouin zone of m-banded Toeplitz operators.* arXiv.2602.09734

Code availability:

[1] dB, Hiltunen: github.com/yannick2305/OpenLimitToeplitz

Contact: yannicd@math.uio.no