Wave localisation in periodic media.

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Floquet transform

- Let $f \in L^2_{\text{loc}}(\mathbb{R}^n)$, the Floquet transform is given by $f(x) \mapsto \mathcal{U}_\Lambda\big[f(x,\alpha)\big] := \sum_{\ell\in\Lambda} f(x-\ell)e^{\mathrm{i}\alpha\cdot\ell}$.

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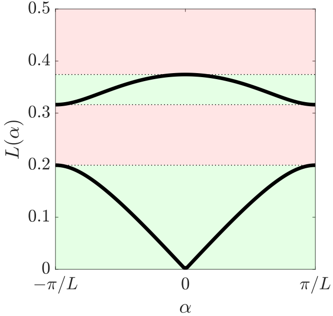
- **Fiber Decomposition:**

$$L = \int_{\mathbb{R}^n/(2\pi\mathbb{Z}^n)}^{\oplus} L(\alpha)$$

- **Spectrum:**

$$\sigma(L) = \bigcup_{\alpha\in\mathbb{R}^n/(2\pi\mathbb{Z}^n)} \sigma\big(L(\alpha)\big)$$

- **Bandgap:** $\rho(L) = \mathbb{R} \setminus \sigma(L)$



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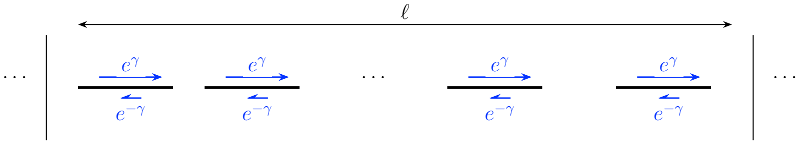
One Dimensional non-Hermitian Helmholtz scattering

- **Model setting:**



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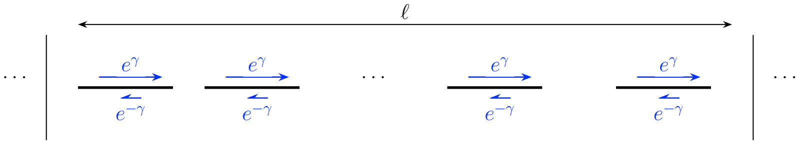

- **Wave equation:**

$$\begin{cases} u''(x) + \gamma u'(x) + \omega^2 u(x) = 0, & x \in D \\ u''(x) + \omega^2 u(x) = 0, & x \in \mathbb{R}\backslash D, \\ u|_+ - u|_- = 0, & \text{on } \partial D, \\ \left.\frac{\partial u}{\partial \nu}\right|_- - \delta \left.\frac{\partial u}{\partial \nu}\right|_+ = 0, & \text{on } \partial D, \\ u(x+\ell) = e^{\mathrm{i}(\alpha + \mathrm{i}\beta)\ell} u(x), & \text{for all } \ell \in \Lambda. \end{cases}$$

  - Imaginary gauge potential $\gamma \in \mathbb{R} \setminus \{0\}$.
  - Contrast parameter $0 < \delta \ll 1$.

Subwavelength resonant frequencies

Definition

Given $\delta > 0$, a subwavelength resonant frequency $\omega = \omega(\delta)$ is defined to be such that

(i) there exists a nontrivial solution to the Helmholtz problem.

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Experimental setup

crystals

MDPI

Article

Soda Cans Metamaterial: A Subwavelength-Scaled Phononic Crystal

Fabrice Lemoult, Nadège Kaina, Mathias Fink and Geoffroy Lerosey *

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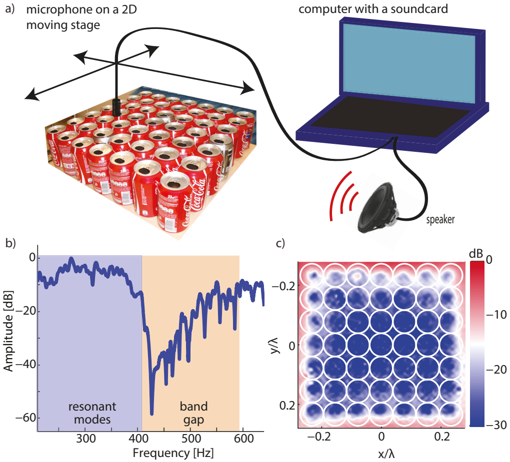

Asymptotic Resonance Expansion

- **Gauge Capacitance matrix**

$$\widehat{C}_{ij}^{\alpha,\beta} = -\frac{l_i}{\int_{D_i} e^{\gamma x} \mathrm{d}x} \left(-e^{\gamma x_j^{\mathsf{L}}} \left.\frac{\mathrm{d}V_i^{\alpha,\beta}}{\mathrm{d}x}\right|_{\mathsf{L}} (x_j^{\mathsf{L}}) + e^{\gamma x_j^{\mathsf{R}}} \left.\frac{\mathrm{d}V_i^{\alpha,\beta}}{\mathrm{d}x}\right|_{\mathsf{R}} (x_j^{\mathsf{R}}) \right),$$

where $V_i^{\alpha,\beta} : \mathbb{R} \to \mathbb{R}$ is known solutions to some ODE.

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where $V_i^{\alpha,\beta} : \mathbb{R} \to \mathbb{R}$ is known solutions to some ODE.

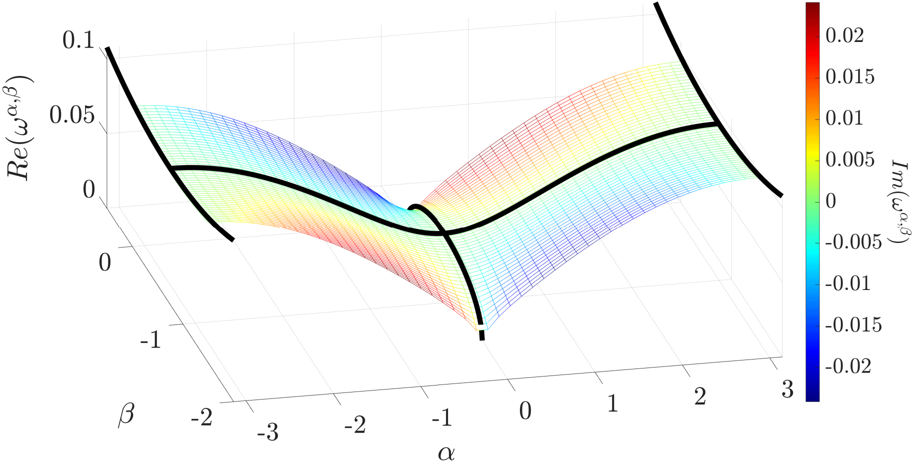
Theorem (dB, Hiltunen)

The k subwavelength (complex) band functions $\alpha + \mathrm{i}\beta \mapsto \omega^{\alpha,\beta}$ satisfy as $\delta \to 0$,

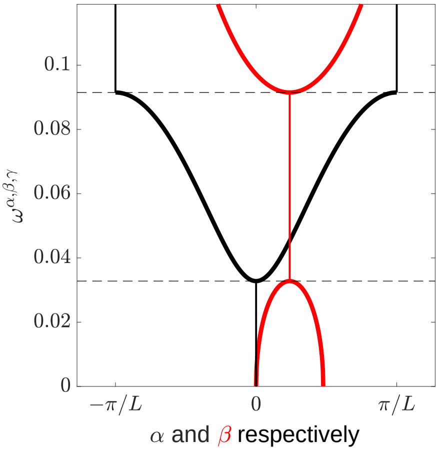
$$\omega_i^{\alpha,\beta} = \sqrt{\delta \lambda_i^{\alpha,\beta}} + \mathcal{O}(\delta),$$

where $(\lambda_i^{\alpha,\beta})_{1 \le i \le k}$ are the eigenvalues of the eigenvalue problem $\widehat{C}_{ij}^{\alpha,\beta} \mathbf{v} = \lambda_i^{\alpha,\beta} \mathbf{v}$.

Complex Band structure for Monomer (k=1) chain



Projected visualisation



Recall from Floquet Theory:

- **In the Spectrum:**
  - β is fixed to $\tilde{\beta}$
  - $\alpha \in Y^*$
  - $$\sigma(\mathcal{C}) = \bigcup_{\alpha \in Y^*} \sigma\left(\widehat{C}_{ij}^{\alpha,\tilde{\beta}}\right)$$
- **In the Bandgap:**
  - $\alpha \in \{0, \pm\pi/L\}$ is fixed to $\tilde{\alpha}$
  - $\beta \in \mathbb{R}$
  - $$\rho(\mathcal{C}) = \bigcup_{\beta \in \mathbb{R}} \sigma\left(\widehat{C}_{ij}^{\tilde{\alpha},\beta}\right)$$

Symbol function and Toeplitz Theory

- Quasiperiodic operator $\widehat{C}^{\alpha,\beta} \in \mathbb{C}^{k\times k}$.
- The real space operator $\mathcal{C} \in \ell^2(\Lambda)$ is a tridiagonal k-Toeplitz operator.
- Discrete Floquet transform $\mathcal{I}: \ L^2(Y) \to \ell^2(\Lambda)$,

$$\mathcal{I}[\psi](\ell) := \frac{1}{|Y^*|}\int_{Y^*} \psi(\alpha)e^{-\mathrm{i}\alpha\cdot\ell}\,\mathrm{d}\alpha$$

- $\mathcal{C}$ and $\widehat{C}^{\alpha,\beta} \in \mathbb{R}^{k\times k}$ are related,

$$\mathcal{C}^{\ell} = \mathcal{I}[\widehat{\mathcal{C}}^{\alpha,\tilde{\beta}}](\ell).$$

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Tridiagonal k-Toeplitz operators.

Example: 2-Toeplitz operator:

$$\mathbf{T}(\mathbf{f}) := \begin{pmatrix} a_1 & b_1 & 0 & 0 & & & \\ c_1 & a_2 & b_2 & 0 & & & \\ 0 & c_2 & a_1 & b_1 & 0 & 0 & \\ 0 & 0 & c_1 & a_2 & b_2 & 0 & \\ & & 0 & c_k & \ddots & & \ddots \\ & & 0 & 0 & & & \\ & & & & \ddots & & \end{pmatrix} = \begin{pmatrix} \mathbf{A}_0 & \mathbf{A}_{-1} & & \\ \mathbf{A}_1 & \mathbf{A}_0 & \mathbf{A}_{-1} & \\ & \mathbf{A}_1 & \ddots & \ddots \\ & & \ddots & \end{pmatrix}$$

- **Symbol function:**

$$f : \mathbb{C} \to \mathbb{C}^{2\times 2}$$

$$z \mapsto \mathbf{A}_{-1} z^{-1} + \mathbf{A}_0 + \mathbf{A}_1 z = \begin{pmatrix} a_1 & b_1 + c_2 z \\ c_1 + b_2 z^{-1} & a_2 \end{pmatrix}.$$

Spectrum of k-Toeplitz operators

- Let us define the sets:

$$\sigma_{\det}(f) := \left\{\lambda \in \mathbb{C} : \det\bigl(f(z) - \lambda\bigr) = 0,\ \exists z \in \mathbb{T}\right\},$$
$$\sigma_{\text{wind}}(f) := \left\{\lambda \in \mathbb{C} \setminus \sigma_{\det}(f) : \text{wind}\bigl(\det\bigl(f(\mathbb{T}) - \lambda\bigr), 0\bigr) \neq 0\right\}.$$

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Theorem (dB, Ammari, Liu, Thalhammer)

Let $f \in \mathbb{C}^{k\times k}(\mathbb{T})$ *be the symbol of a tridiagonal* k*-Toeplitz operator* $T(f)$ *it holds that*

$$\sigma_{\text{det}}(f) \cup \sigma_{\text{wind}}(f) \subseteq \sigma\big(T(f)\big) \subseteq \sigma_{\text{det}}(f) \cup \sigma_{\text{wind}}(f) \cup \sigma(\mathbf{B}_0),$$

$\mathbf{B_0} \in \mathbb{R}^{k-1\times k-1}$ *is the principal minor of* $\mathbf{A}_0$.

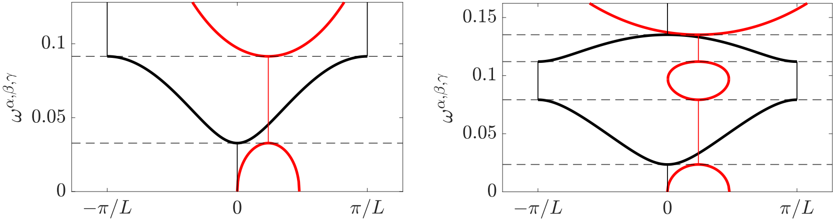
Eigenvector Asymptotics

Lemma (dB, Hiltunen)

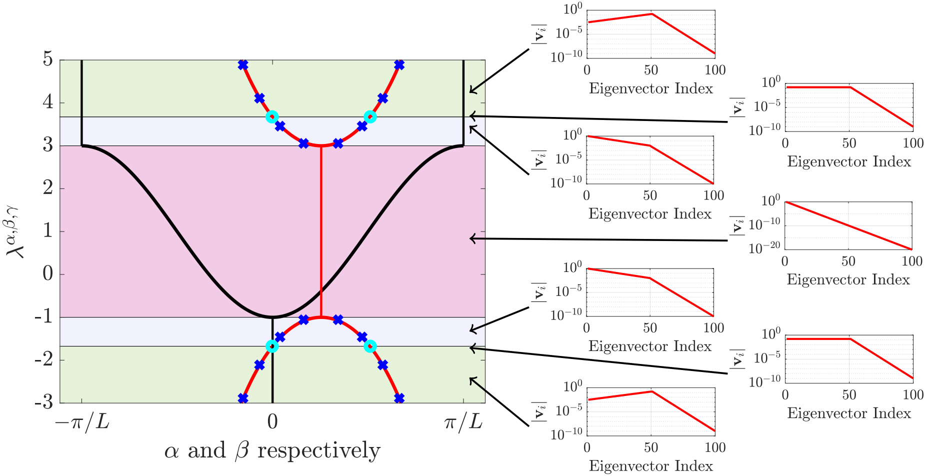
Let $\lambda \in \mathbb{R}$ *be an eigenfrequency, such that* $\mathbf{T}(f)\mathbf{u} = \lambda \mathbf{u}$ *then*

$$\frac{|\mathbf{u}^{(i+k)}|}{|\mathbf{u}^{(i)}|} = \mathcal{O}\left(e^{-(r-\tilde{\beta})}\right),$$

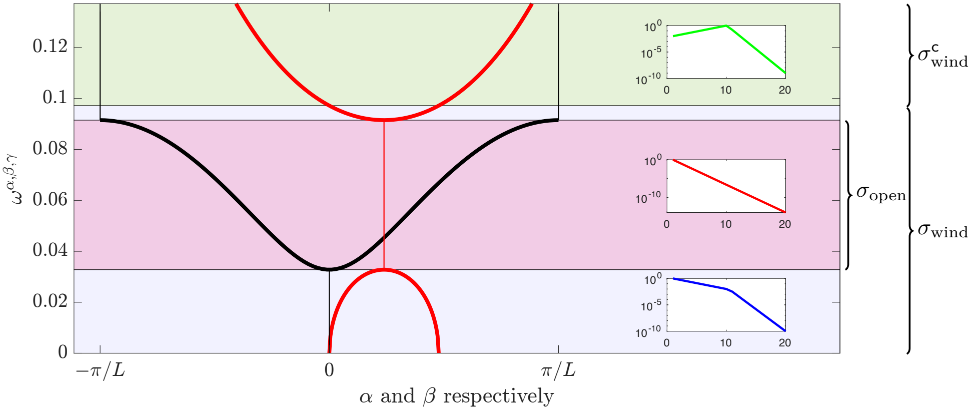
where $r = \frac{1}{2}\log\left(\prod_{i=1}^{k} \frac{b_i}{c_i}\right)$ *is fixed and* $\tilde{\beta}(\lambda) = \operatorname{arccosh}\left(-\frac{g(\lambda)}{2Ae^r}\right)$*, for* $A = (-1)^{k+1}\prod_{i=1}^{k} c_i$*, and* $g(\lambda)$ *a known polynomial.*



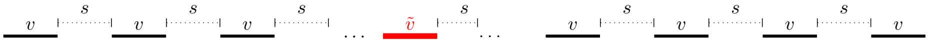
Defect-Induced Localization Transitions

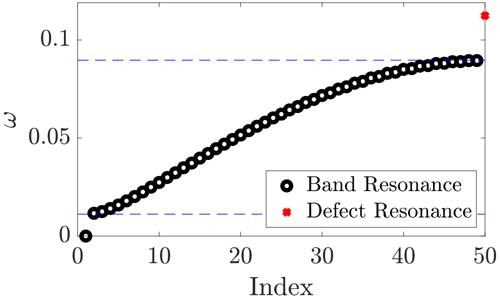


Topological properties

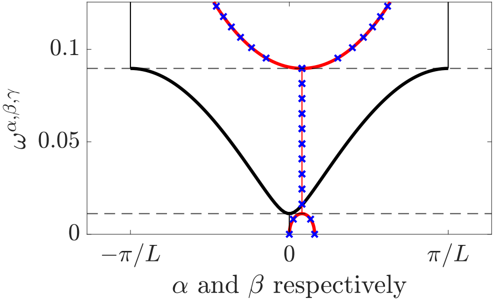


- **Changed wave speed:**





(c) Resonances



(d) Exponential decay vs Complex band structure

Finite Media and Pseudospectra

- **Question:**
  - Why is the Complex Band Structure valid for finite media?

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Definition

λ is ε-pseudoeigenvalue of $\mathbf{A}$ with ε-pseudoeigenvector if,

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- **Solution:**
  - Truncated eigenvectors of a semi-infinite system become ε_N-pseudoeigenvectors in the finite system

$$\varepsilon_N = e^{-\beta(\lambda)N}.$$

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Two-Dimensional Crystal

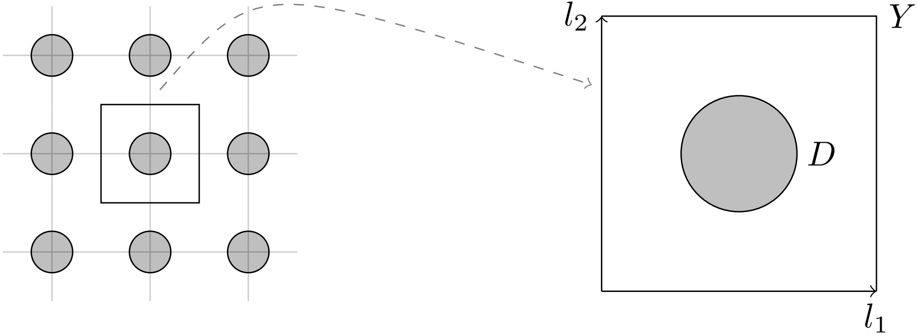

Figure: Square lattice with a single resonator inside the unit cell ($N = 1$).

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Bandgap Green's function

- Two-dimensional Helmholtz problem

$$
\begin{cases}
\Delta u + k^2 u = 0, & \text{in } Y \setminus \overline{D}, \\
\Delta u + k_i^2 u = 0, & \text{in } D_i, \\
u|_+ - u|_- = 0, & \text{on } \partial D, \\
\left.\dfrac{\partial u}{\partial \nu}\right|_- - \delta \left.\dfrac{\partial u}{\partial \nu}\right|_+ = 0, & \text{on } \partial D, \\
u(x+\ell) = e^{\mathrm{i}(\alpha + \mathrm{i}\beta)\cdot \ell} u(x), & \text{for all } \ell \in \Lambda.
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- For a real quasimomentum ($\beta = 0$), the α-quasiperiodic Green's function $G^{\alpha,k}$ satisfies

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and is given by

$$G^{\alpha,k}(x) = \frac{1}{|Y|} \sum_{q\in\Lambda^*} \frac{e^{\mathrm{i}(\alpha+q)\cdot x}}{k^2 - |\alpha+q|^2}.$$

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- The band gap Green's function satisfies

$$\Delta\tilde{G}^{\alpha,\beta,k}(x) - 2\beta\cdot\nabla\tilde{G}^{\alpha,\beta,k}(x) + (k^2+|\beta|^2)\tilde{G}^{\alpha,\beta,k}(x) = \sum_{m\in\Lambda} e^{\mathrm{i}\alpha\cdot m}\delta(x-m).$$

Generalised Green's function

- Poisson summation formula,

$$\tilde{G}^{\alpha,\beta,k}(x) = \frac{1}{|Y|} \sum_{q \in \Lambda^*} \frac{e^{\mathrm{i}(\alpha+q)\cdot x}}{k^2 + |\beta|^2 - 2\mathrm{i}\beta \cdot (\alpha + q) - |\alpha + q|^2}.$$

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- **Solution:** Accelerated lattice sum methods have been introduced.

Layer Potential Techniques

- Let $D \subset Y$ be a domain in $\mathbb{R}^2$ with a boundary $\partial D \in C^{1,s}$, with some $0 < s < 1$.

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- Single Layer Potential of ϕ,

$$\begin{aligned}\tilde{\mathcal{S}}_{D_i}^{\alpha,\beta,k} : L^2(\partial D) &\to H^1(\partial D)\\ \phi &\mapsto \tilde{\mathcal{S}}_{D}^{\alpha,\beta,k}[\phi](x) = \int_{\partial D} \tilde{G}^{\alpha,\beta,k}(x-y)\phi(y)\mathrm{d}\sigma(y), \quad x \in \mathbb{R}^2.\end{aligned}$$

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$$\phi \mapsto \tilde{\mathcal{S}}_{D}^{\alpha,\beta,k}[\phi](x) = \int_{\partial D} \tilde{G}^{\alpha,\beta,k}(x-y)\phi(y)\mathrm{d}\sigma(y), \quad x \in \mathbb{R}^2.$$

- Field solution

$$v(x) = \begin{cases} \tilde{\mathcal{S}}_{D}^{\alpha,\beta,k}[\phi](x), & \text{in } Y \setminus \overline{D}, \\ \tilde{\mathcal{S}}_{D_i}^{\alpha,\beta,k_i}[\psi](x), & \text{in } D_i, \end{cases}$$

and unknown $\phi, \psi \in L^2(\partial D)$.

Layer Potential Techniques

- Let $D \subset Y$ be a domain in $\mathbb{R}^2$ with a boundary $\partial D \in C^{1,s}$, with some $0 < s < 1$.
- Single Layer Potential of ϕ,

$$\tilde{\mathcal{S}}_{D_i}^{\alpha,\beta,k} : L^2(\partial D) \to H^1(\partial D)$$
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- Gap mode

$$u(x) = e^{-\beta \cdot x} v(x).$$

Multipole Expansion Method

- Circular Resonators $\mapsto$ Fourier Basis $\{e^{\mathrm{i}n\theta}\}_{n\in\mathbb{Z}}$ of $L^2(\partial D)$.

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$$
\begin{aligned}
(\tilde{\mathcal{S}}_D^{\alpha,\beta,k})_{m,n} &= \frac{1}{2\pi}\left\langle e^{\mathrm{i}m\theta}, \tilde{\mathcal{S}}_D^{\alpha,\beta,k}[e^{\mathrm{i}n\varphi}]\right\rangle \quad \in \mathbb{C}^{(2M+1)\times(2M+1)} \\
&= \frac{r}{2\pi}\int_0^{2\pi} e^{-\mathrm{i}m\theta}\int_0^{2\pi} \tilde{G}^{\alpha,\beta,k}\big(r(e^{\mathrm{i}\theta} - e^{\mathrm{i}\varphi})\big)e^{\mathrm{i}n\varphi}\,\mathrm{d}\varphi\,\mathrm{d}\theta.
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\end{aligned}
$$

- Replace $\tilde{G}^{\alpha,\beta,k}$ with preferred (accelerated) representation.
- Evaluating the boundary integral is a numerically delicate task
  - Bessel representation if possible.
  - Trapezoidal/Simpsons rule.
  - Adaptive Quadrature.

Complex-Quasiperiodic Capacitance

Theorem (dB & Hiltunen)

Consider a system of N subwavelength resonators in the unit cell Y and assume $\tilde{G}^{\alpha,\beta,k}(x)$ is well-defined and $\tilde{\mathcal{S}}_D^{\alpha,\beta,0}$ is invertible. As $\delta \to 0$, the subwavelength resonant frequencies $\omega_n^{\alpha,\beta}$ satisfy the asymptotic formula

$$\omega^{\alpha,\beta} = \sqrt{\delta \lambda_n^{\alpha,\beta}} + \mathcal{O}(\delta), \quad n = 1, \dots, N,$$

where $\{\lambda_n^{\alpha,\beta}\}$ are the N eigenvalues of the generalised capacitance matrix $\mathcal{C}^{\alpha,\beta} \in \mathbb{C}^{N\times N}$, given by

$$\mathcal{C}_{ij}^{\alpha,\beta} = -\frac{v_i^2}{|D_i|} \int_{D_i} e^{-\mathrm{i}\beta\cdot x} \psi_j \mathrm{d}\sigma, \quad \psi_i = (\tilde{\mathcal{S}}_D^{\alpha,\beta,0})^{-1}[e^{\beta\cdot x}\chi_{\partial D_i}].$$

Spectral Plot and complex Band Functions

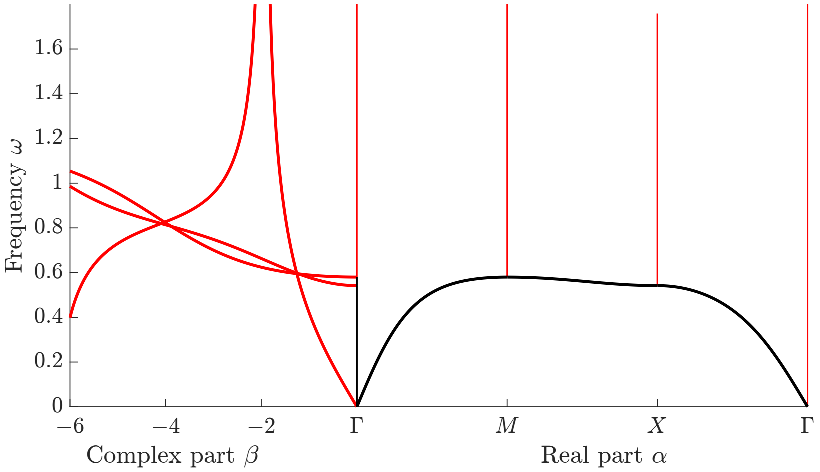

Figure: Generate the spectral plot in 0.5s and an expected error of $\mathcal{O}(10^{-3})$.

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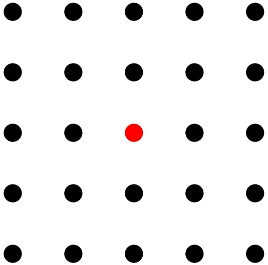
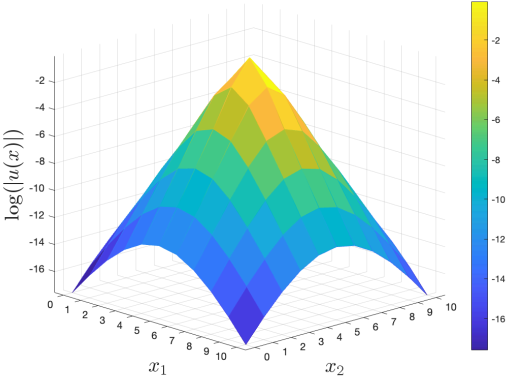
2D Defect Modes



(a) Finite defected resonator lattice.



(b) Defect eigenmode.

Defect Modes and Spectral Plot

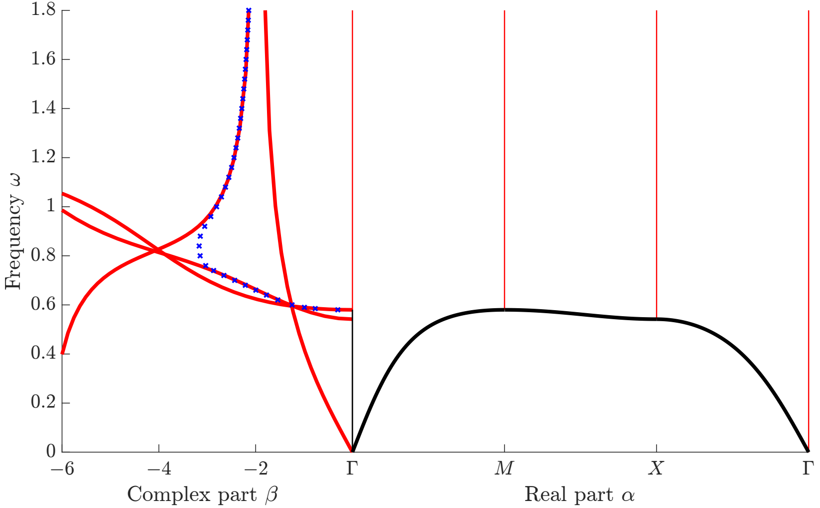


Figure: Decay for β horizontal ($\rightarrow$).

Complex Floquet transform

Theorem (dB & Hiltunen)

Let u *be a complex Bloch mode for some* $\beta \in \mathbb{R}^n \setminus \{0\}$*, i.e.,*

$$\big|u(x+\ell)\big| = e^{\beta\cdot\ell}\big|u(x)\big|, \quad \forall \ell \in \Lambda.$$

Then the complex Floquet transform is well-defined and given by,

$$u(x) \mapsto \mathcal{U}_\Lambda\big[u(x,\alpha+\mathrm{i}\beta)\big] := \sum_{\ell\in\Lambda} u(x-\ell)e^{\mathrm{i}(\alpha+\mathrm{i}\beta)\cdot\ell}.$$

Truncated complex Floquet transform

- Defect mode $\mathbf{u}_m$, with decay length $\tilde{\beta}$.

Truncated complex Floquet transform

- Defect mode $\mathbf{u}_m$, with decay length $\tilde{\beta}$.
- Finite lattice Λ_t,

$$\|(\hat{u})_\alpha\|_2 := \left\| \sum_{m \in \Lambda_t} \mathbf{u}_m e^{\mathrm{i}(\alpha + \mathrm{i}\tilde{\beta}) \cdot m} \right\|_2 .$$

Truncated complex Floquet transform

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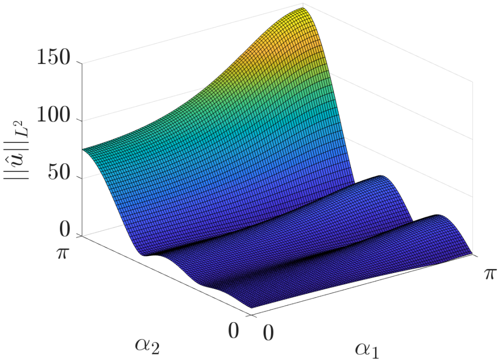

Figure: $\|(\hat{u})_\alpha\|_2$ at gap frequency $\omega = 0.6$.

Truncated complex Floquet transform

- Defect mode $\mathbf{u}_m$, with decay length $\tilde{\beta}$.
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$$\|(\hat{u})_\alpha\|_2 := \left\| \sum_{m\in\Lambda_t} \mathbf{u}_m e^{\mathrm{i}(\alpha+\mathrm{i}\tilde{\beta})\cdot m} \right\|_2 .$$

- $\|(\hat{u})_\alpha\|_2$ has distinct peaks,

$$\alpha = \max_{\alpha\in Y^*} \|(\hat{u})_\alpha\|_2.$$

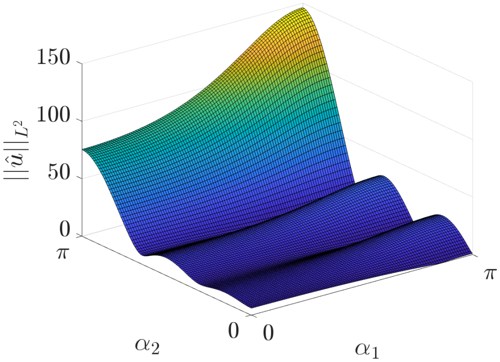
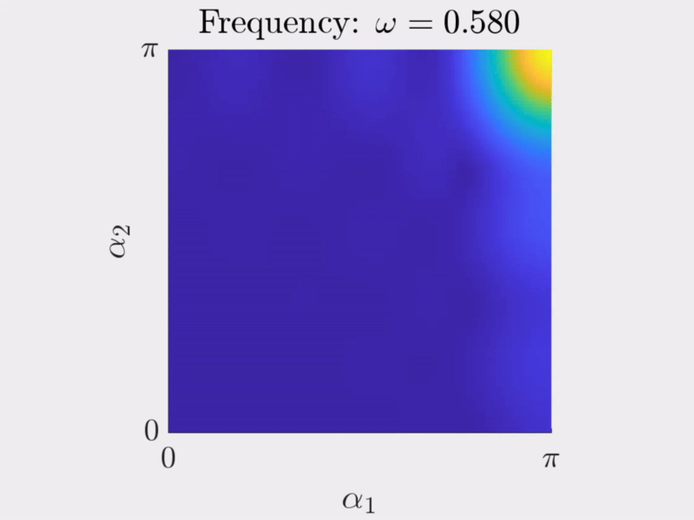
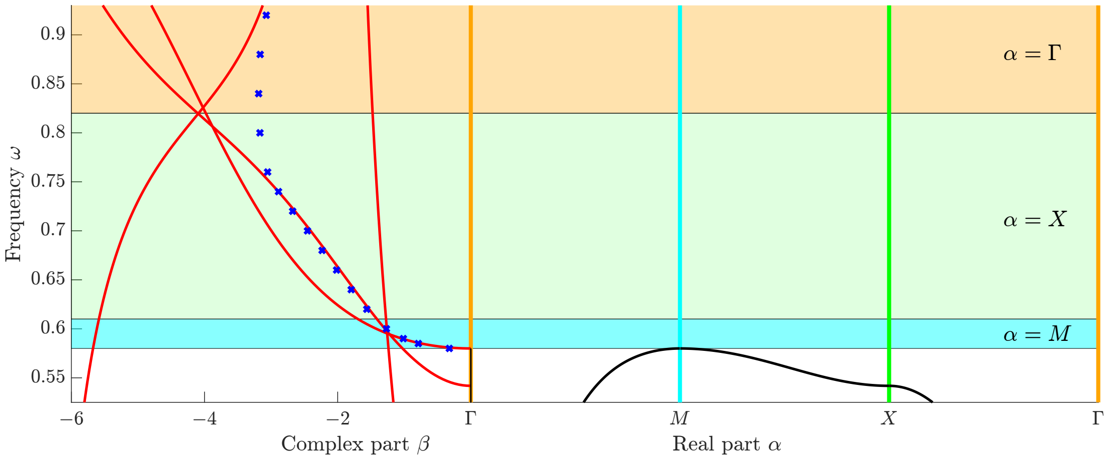


Figure: $\|(\hat{u})_\alpha\|_2$ at gap frequency $\omega = 0.6$.

Phase-shift within the Band Gap



3 Phase-shift zones



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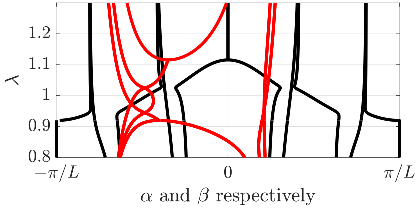
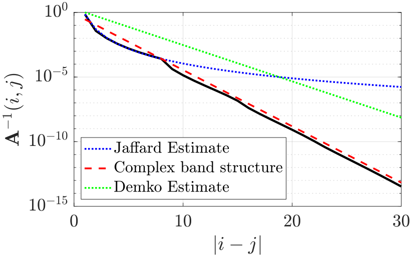
- **3D resonator chains:**

- **Dense Toeplitz operators:**

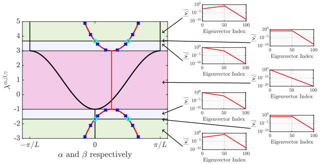
$$\mathbf{A} = \begin{pmatrix} a_0 & a_{-1} & a_{-2} & a_{-3} & \cdots \\ a_1 & a_0 & a_{-1} & a_{-2} & \cdots \\ a_2 & a_1 & a_0 & a_{-1} & \cdots \\ a_3 & a_2 & a_1 & a_0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

- **Long term:**
  - Electromagnetic Resonances (Maxwell).
  - Finite Element methods.

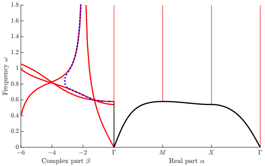




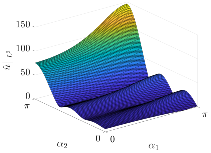
Questions?



(a) $1D$ Decay Transition.



(b) $2D$ Spectral Plot.



(c) Floquet Transform.

References and Code availability:

[1] dB, Ammari, Thalhammer, Liu, Barandun: *Spectra and pseudo-spectra of tridiagonal k-Toeplitz matrices and the topological origin of the non-Hermitian skin effect* 10.1088/1751-8121/add5ab
[2] dB, Hiltunen: *Complex Band Structure for Subwavelength Evanescent Waves* 10.1111/sapm.70022
[3] dB, Hiltunen: *Complex Brillouin Zone for Localised Modes in Hermitian and Non-Hermitian Problem* arXiv.2502.06620
[4] dB, Hiltunen: *Complex Band Structure and localisation transition for tridiagonal non-Hermitian k-Toeplitz operators with defects* arXiv.2505.23610
[5] dB, Hiltunen: *Github: PhotonicBandGaps* github.com/yannick2305/PhotonicBandGaps
[6] dB, Hiltunen: *Github: Non-Hermitian Localisation* github.com/yannick2305/Non-Hermitian-Localisation

Contact: yannicd@math.uio.no

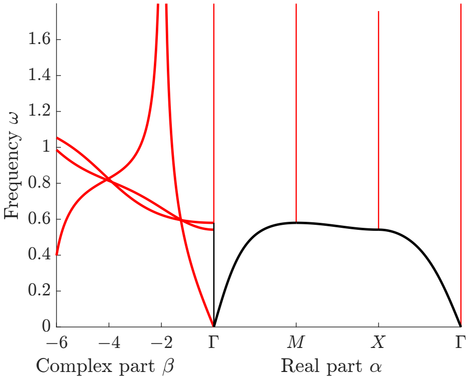
What does the spectral Plot mean?

- Band formulation, i.e. $\beta = 0$,

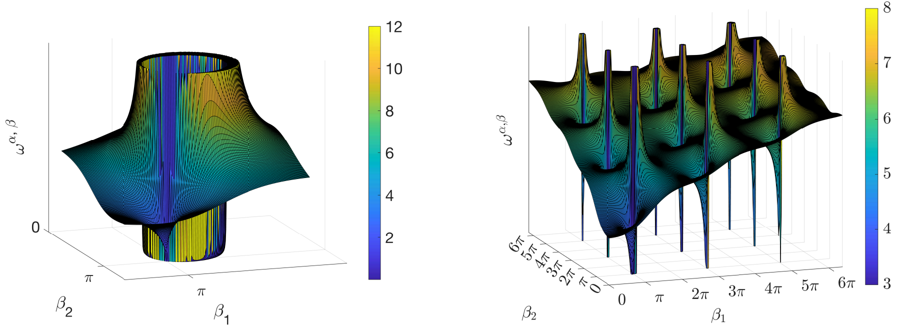
$$\begin{cases} \Delta u + k^2 u = 0, & \text{in } Y \setminus \overline{D}, \\ \Delta u + k_i^2 u = 0, & \text{in } D_i, \\ u|_+ - u|_- = 0, & \text{on } \partial D, \\ \left.\frac{\partial u}{\partial \nu}\right|_- - \delta \left.\frac{\partial u}{\partial \nu}\right|_+ = 0, & \text{on } \partial D, \\ u(x+\ell) = e^{\mathrm{i}(\alpha + \mathrm{i}\beta)\cdot \ell} u(x), & \text{for all } \ell \in \Lambda. \end{cases}$$

- Gap formulation i.e. $\beta \neq 0$, set $v(x) := e^{\beta \cdot x} u(x)$,

$$\begin{cases} \Delta v - 2\beta \cdot \nabla v + (k^2 + |\beta|^2) v = 0, & \text{in } Y \setminus \overline{D}, \\ \Delta v - 2\beta \cdot \nabla v + (k_i^2 + |\beta|^2) v = 0, & \text{in } D_i, \\ v|_+ - v|_- = 0, & \text{on } \partial D, \\ \left.\frac{\partial v}{\partial \nu}\right|_- - (\beta \cdot \nu) v - \delta \left(\left.\frac{\partial v}{\partial \nu}\right|_+ - (\beta \cdot \nu) v \right) = 0, & \text{on } \partial D, \\ v(x+\ell) = e^{\mathrm{i}\alpha \cdot \ell} v(x), & \text{for all } \ell \in \Lambda. \end{cases}$$



Motivation: Singularities in the Band functions



(d) Close View. (e) Wide View.

Figure: Band function $\omega^{\alpha,\beta}$ for fixed $\alpha = [\pi, \pi]$ and Resonator radius $R = 0.005$.